Assignment # 7	Due on December 18th in class

1. Prove that $B(\mathbb{C}^n, \mathbb{C})$ is isomorphic (as a normed space) to $\mathbb{C}^n$. To be more precise, you need to show that there exists a map

$$\rho\colon B(\mathbb{C}^n, \mathbb{C}) \longrightarrow \mathbb{C}^n,$$

such that

i) ρ is bijective (both injective and surjective)

ii) ρ is linear. In other words, for all $x, y \in B(\mathbb{C}^n, \mathbb{C})$ and $\lambda \in \mathbb{C}$, we have

$$\rho(x+y) = \rho(x) + \rho(y) \text{ and } \rho(\lambda x) = \lambda \rho(x).$$

iii) ρ preserves the norm structure. In other words, for all $x \in B(\mathbb{C}^n, \mathbb{C})$, we have $\|\rho(x)\| = \|x\|$.

2. Assuming we have the following fact (which can be derived from the Hahn-Banach Theorem):

“ Let X be a Banach space. Then for any $x \in X$ with $x \neq 0$, there exists a bounded linear functional on f on X (in other words, $f \in B(X, \mathbb{C})$) such that $f(x) \neq 0$. ”

Consider the following map

$$\rho\colon X \longrightarrow X^{**},\ x \mapsto \rho(x) \text{ with } \rho(x)(f) = f(x)\ \forall f \in X^*.$$

Prove that the map ρ is linear and injective.

Solution:

1.

Proof. Let $e_1, \cdots, e_n$ be an orthogonal basis for $\mathbb{C}^n$. Define

$$\rho\colon B(\mathbb{C}^n, \mathbb{C}) \longrightarrow \mathbb{C}^n,\ x \mapsto (x(e_1), \cdots, x(e_n))$$

It is not hard to check that ρ is bijective and linear.

To check that ρ preserve the norm, just need to apply the classical Cauchy-Schwarz inequality.

$\square$

2.

Proof. Easy to check that ρ is linear.

Now we try to show that ρ is injective. Suppose not, then there exists $x, y \in X$, such that $\rho(x) = \rho(y)$. In other words, for any $f \in X^*$, we have

$$\rho(x)(f) = \rho(y)(f).$$

It follows that

$$f(x) = f(y) \ \ \forall f \in X^*.$$

In other words,

$$f(x - y) = 0 \ \ \forall f \in X^*,$$

which contradicts the fact in the statement.

$\square$